

SIGNATURES OF HERMITIAN FORMS AND “PRIME IDEALS” OF WITT GROUPS

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ABSTRACT. It is shown that H -signatures define canonical invariants of hermitian forms over algebras with involution, defined over formally real fields, by means of the study of a natural generalization of the prime spectrum of Witt rings to the setting of Witt groups of such forms.

1. INTRODUCTION

Let F be a formally real field and let (A, σ) be an F -algebra with involution. In [1] the notion of H -signature $\text{sign}_P^H h$ of a hermitian form h over (A, σ) with respect to an ordering P on F is defined. This is a refinement of the definition of signature $\text{sign}_P h$ in [3]. Both signatures are defined via scalar extension to a real closure F_P of F at P , followed by an application of Morita theory, which reduces the computation to what essentially is the case of classical signatures of quadratic forms.

Pfister’s local-global principle holds for both signatures since $\text{sign}_P h = 0$ if and only if $\text{sign}_P^H h = 0$, see [11], and the Knebusch trace formula holds for H -signatures, see [1].

Both sign_P^H and sign_P should be considered as relative invariants of forms in the sense that knowledge about the non-triviality of the signature of some fixed reference form(s) is needed in order to compute the signature of an arbitrary form. In [3] this reference form is taken to be the unit form $\langle 1 \rangle_\sigma$. As was demonstrated in [1, 3.11], $\text{sign}_P \langle 1 \rangle_\sigma = 0$ whenever σ becomes hyperbolic over the extended algebra $(A \otimes_F F_P, \sigma \otimes \text{id})$ and thus $\langle 1 \rangle_\sigma$ can then not be used as a reference form in this case. In [1, 6.4] the existence of a finite tuple H of reference forms was established that does not suffer from this problem.

In this paper we show that the tuple H can be replaced by just one reference form that has non-zero H -signature with respect to all relevant orderings on F . The remainder of the paper consists of studying the “prime ideals” of the Witt group $W(A, \sigma)$, obtaining results that are analogues of the classification of prime ideals of the Witt ring $W(F)$ by Harrison [5] and Lorenz-Leicht [13]. As a consequence we show that H -signatures are a canonical notion in the sense that they can be identified with a natural set of morphisms from $W(A, \sigma)$ to \mathbb{Z} .

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2. NOTATION

We give a brief overview of the notation we use and refer to the standard references [7], [8], [9] and [16] for the details.

Let F be a field of characteristic different from 2. We write $W(F)$ for the *Witt ring* of Witt equivalence classes of quadratic forms over F and $I(F)$ for its *fundamental ideal*.

An F -*algebra with involution* is a pair (A, σ) where A is a finite-dimensional F -algebra, whose centre $Z(A)$ satisfies $[Z(A) : F] =: \kappa \leq 2$, and which is assumed to be either simple (if $Z(A)$ is a field) or a direct product of two simple algebras (if $Z(A) = F \times F$) and $\sigma : A \rightarrow A$ is an F -linear involution. Let $\text{Sym}(A, \sigma) = \{a \in A \mid \sigma(a) = a\}$ and $\text{Sym}(A, \sigma)^\times = \text{Sym}(A, \sigma) \cap A^\times$. When $\kappa = 1$, we say that σ is *of the first kind*. When $\kappa = 2$, we say that σ is *of the second kind* (or *unitary*). Assume $\kappa = 1$ and $\dim_F A = m^2 \in \mathbb{N}$. Then σ is *orthogonal* or *symplectic* if $\dim_F \text{Sym}(A, \sigma) = m(m+1)/2$ or $m(m-1)/2$, respectively.

For $\varepsilon \in \{-1, 1\}$ we write $W_\varepsilon(A, \sigma)$ for the *Witt group* of Witt equivalence classes of ε -hermitian forms $h : M \times M \rightarrow A$, defined on finitely generated right A -modules M . All forms in this paper are assumed to be non-singular and are identified with their Witt equivalence classes, unless indicated otherwise. We write $+$ for the sum in both $W(F)$ and $W_\varepsilon(A, \sigma)$. The group $W_\varepsilon(A, \sigma)$ is a $W(F)$ -module and we denote the product of $q \in W(F)$ and $h \in W_\varepsilon(A, \sigma)$ by $q \cdot h$.

From the structure theory of F -algebras with involution it follows that A is isomorphic to a full matrix algebra $M_n(D)$ for a unique $n \in \mathbb{N}$ and an F -division algebra D (unique up to isomorphism), equipped with an involution ϑ of the same kind as σ . Then (A, σ) and (D, ϑ) are Morita equivalent, which yields a (non-canonical) isomorphism $W_\varepsilon(A, \sigma) \cong W_{\varepsilon\mu}(D, \vartheta)$ where $\mu \in \{-1, 1\}$. For the purpose of this paper (the study of signatures) and without loss of generality we may assume that $\varepsilon = \mu = 1$, cf. [1, 2.1] and thus only consider hermitian forms over (A, σ) .

Let $h : M \times M \rightarrow A$ be a hermitian form over (A, σ) . By Wedderburn theory, M decomposes into a direct sum of simple right A -modules that are all isomorphic to D^n , $M \cong (D^n)^k$, for some $k \in \mathbb{N}$ which we call the *rank* of h , denoted $\text{rank } h$. The rank of h is invariant under Morita equivalence, cf. [2, 2.1] or [4, §3.2.1].

For $\ell \in \mathbb{N}$, diagonal forms on the free A -module A^ℓ are denoted by $\langle a_1, \dots, a_\ell \rangle_\sigma$ with $a_1, \dots, a_\ell \in \text{Sym}(A, \sigma)^\times$. We call ℓ the *dimension* of the form. Note that the rank of this form is equal to ℓn and that its dimension may not be invariant under Morita equivalence. If A is a division algebra, all hermitian forms can be expressed up to isometry in diagonal form and rank and dimension are equal.

Assume now that F is a formally real field with space of orderings X_F . For $P \in X_F$ the H -signature of hermitian forms over (A, σ) induces a morphism of additive groups $\text{sign}_P^H : W(A, \sigma) \rightarrow \mathbb{Z}$ that respects the $W(F)$ -module structure of $W(A, \sigma)$, cf. [1, 3.6]. The set of *nil-orderings* is defined by $\text{Nil}[A, \sigma] := \{P \in X_F \mid \text{sign}_P^H \equiv 0\}$ and only depends on the Brauer class of A and the type of σ (orthogonal, symplectic or unitary). We refer to [1] for the precise definitions.

3. A REFERENCE FORM WITH POSITIVE SIGNATURE

Lemma 3.1. *Let h_1, \dots, h_ℓ be hermitian forms over (A, σ) and let*

$$V = \{P \in X_F \mid \exists i \in \{1, \dots, \ell\} \text{ such that } \text{sign}_P^H h_i \neq 0\}.$$

Then there exists a hermitian form h over (A, σ) such that

$$V = \{P \in X_F \mid \text{sign}_P^H h \neq 0\}.$$

Proof. By induction on ℓ . The case $\ell = 1$ is clear. Consider the case ℓ and let

$$V' = \{P \in X_F \mid \exists i \in \{1, \dots, \ell - 1\} \text{ such that } \text{sign}_P^H h_i \neq 0\}.$$

By induction there exists a hermitian form h' such that $V' = \{P \in X_F \mid \text{sign}_P^H h' \neq 0\}$. Note that V' is clopen in X_F since the total signature map $\text{sign}^H h'$ is continuous [1, 7.2]. Thus, by [9, VIII, 6.10] there exists a quadratic form q over F such that $\{P \in X_F \mid \text{sign}_P q = 0\} = V'$. Let

$$h := h' + q \cdot h_\ell.$$

Let $P \in V$. We have to consider two cases:

- $P \in V'$. Then $\text{sign}_P^H h' \neq 0$ while $\text{sign}_P q = 0$. Thus $\text{sign}_P^H h \neq 0$.
- $P \in V \setminus V'$. Then $\text{sign}_P^H h' = 0$ and $\text{sign}_P q \neq 0$. We also have $\text{sign}_P^H h_\ell \neq 0$ by definition of V . Therefore $\text{sign}_P^H h \neq 0$.

We conclude that $V \subseteq \{P \in X_F \mid \text{sign}_P^H h \neq 0\}$. For the reverse inclusion, let $P \in X_F$ be such that $\text{sign}_P^H h \neq 0$. If $\text{sign}_P^H h' \neq 0$, then $P \in V' \subseteq V$. If $\text{sign}_P^H h' = 0$ then $\text{sign}_P^H (q \cdot h_\ell) \neq 0$, which implies $\text{sign}_P^H h_\ell \neq 0$ and thus $P \in V$. ■

Proposition 3.2. *There exists a hermitian form h over (A, σ) such that $\text{sign}_P^H h \neq 0$ for every $P \in X_F \setminus \text{Nil}[A, \sigma]$.*

Proof. Let $H = (h_1, \dots, h_\ell)$. For every $P \in X_F \setminus \text{Nil}[A, \sigma]$ one of $\text{sign}_P^H h_1, \dots, \text{sign}_P^H h_\ell$ is non-zero. Therefore

$$X_F \setminus \text{Nil}[A, \sigma] = \{P \in X_F \mid \exists i \in \{1, \dots, \ell\} \text{ such that } \text{sign}_P^H h_i \neq 0\}$$

and we conclude by (3.1). ■

In light of (3.2) we extend the definition of H -signature [1, 3.9] to allow tuples $H = (h_1, \dots, h_\ell)$ where the h_i are not necessarily of dimension one. In [1, 6.4] we show that the forms h_1, \dots, h_ℓ can be chosen to be diagonal. Thus the form h from (3.2) can also be chosen to be diagonal, which can be seen from the proof of (3.1). It also follows from (3.2) that we can replace the whole tuple H in the definition of sign^H by the single form h , thus trading a tuple of multiple forms of dimension one for a single form of possibly large dimension. An immediate consequence is then:

Corollary 3.3. *Let h be as in (3.2), then $\text{sign}_P^{(h)} h > 0$ for every $P \in X_F \setminus \text{Nil}[A, \sigma]$.*

4. M-IDEALS OF MODULES

Let (A, σ) be an F -algebra with involution. We want to mimic the well-known classification of all prime ideals of $W(F)$ in terms of the kernels of the signature maps. We take some moments to look at the properties of $\ker \text{sign}_P^H$ for $P \in X_F$, inside $W(A, \sigma)$ considered as a $W(F)$ -module. It follows from [1, 3.6] that

- (1) $(\ker \text{sign}_P) \cdot W(A, \sigma) \subseteq \ker \text{sign}_P^H$;
- (2) $W(F) \cdot \ker \text{sign}_P^H \subseteq \ker \text{sign}_P^H$;
- (3) for every $q \in W(F)$ and every $h \in W(A, \sigma)$, $q \cdot h \in \ker \text{sign}_P^H$ implies that $q \in \ker \text{sign}_P$ or $h \in \ker \text{sign}_P^H$.

These properties motivate the following

Definition 4.1. Let R be a commutative ring and let M be an R -module. An m -ideal of M is a pair (I, N) where I is an ideal of R , N is a submodule of M , and such that $I \cdot M \subseteq N$.

An m -ideal (I, N) of M is *prime* if I is a prime ideal of R (we assume that all prime ideals are proper), N is a proper submodule of M , and for every $r \in R$ and $m \in M$, $r \cdot m \in N$ implies that $r \in I$ or $m \in N$.

Example 4.2. The pair $(\ker \text{sign}_P, \ker \text{sign}_P^H)$ is a prime m -ideal of the $W(F)$ -module $W(A, \sigma)$ whenever $P \in X_F \setminus \text{Nil}[A, \sigma]$.

The following lemma is immediate.

Lemma 4.3. Let (I, N) be a prime m -ideal of the R -module M . Then $I = \{r \in R \mid rM \subseteq N\}$.

Definition 4.4. Let R and S be commutative rings, let M be an R -module and N an S -module. We say that a pair (φ, ψ) is an (R, S) -morphism (of modules) from M to N if

- (1) $\varphi : R \rightarrow S$ is a morphism of rings (and in particular $\varphi(1) = 1$);
- (2) $\psi : M \rightarrow N$ is a morphism of additive groups;
- (3) for every $r \in R$ and $m \in M$, $\psi(r \cdot m) = \varphi(r) \cdot \psi(m)$.

We call an (R, S) -morphism (φ, ψ) *trivial* if $\psi = 0$. We denote the set of all (R, S) -morphisms from M to N by $\text{Hom}_{(R, S)}(M, N)$ and its subset of non-trivial (R, S) -morphisms by $\text{Hom}_{(R, S)}^*(M, N)$.

Example 4.5. The pair $(\text{sign}_P, \text{sign}_P^H)$ is a $(W(F), \mathbb{Z})$ -morphism from $W(A, \sigma)$ to \mathbb{Z} and is trivial if and only if $P \in \text{Nil}[A, \sigma]$.

The following three lemmas are immediate.

Lemma 4.6. Let (φ, ψ) be an (R, S) -morphism from M to N . Then $(\ker \varphi, \ker \psi)$ is an m -ideal of the R -module M . Furthermore, $(\ker \varphi, \ker \psi)$ is prime if and only if the $\varphi(R)$ -module $\psi(M)$ is torsion-free.

Lemma 4.7. Let (I, N) be an m -ideal of the R -module M . Then M/N is canonically an R/I -module, where the product is defined by $(r + I) \cdot (m + N) = r \cdot m + N$.

Lemma 4.8. *Let (I, N) be an m -ideal of the R -module M , and let $\varphi : R \rightarrow R/I$ and $\psi : M \rightarrow M/N$ be the canonical projections. Then (φ, ψ) is an $(R, R/I)$ -morphism of modules. Furthermore, (I, N) is prime if and only if $R/I \neq \{0\}$ and M/N is a non-zero torsion-free R/I -module.*

Definition 4.9. Let (φ, ψ_1) and (φ, ψ_2) be (R, S) -morphisms of modules from M to N_1 and N_2 respectively. We say that (φ, ψ_1) and (φ, ψ_2) are *equivalent* if there is an isomorphism of $\text{Im } \varphi$ -modules $\vartheta : \text{Im } \psi_1 \rightarrow \text{Im } \psi_2$ such that $\psi_2 = \vartheta \circ \psi_1$.

5. CLASSIFICATION OF PRIME m -IDEALS OF $W(A, \sigma)$

Let (I, N) be a prime m -ideal of the $W(F)$ -module $W(A, \sigma)$. By classical results of Harrison [5] and Lorenz-Leicht [13] on $W(F)$, cf. [9, VIII, 7.5], there are only three possibilities for I :

- (1) $I = \ker \text{sign}_P$ for some $P \in X_F$;
- (2) $I = \ker(\pi_p \circ \text{sign}_p)$ for some $P \in X_F$ and prime $p > 2$, where $\pi_p : \mathbb{Z} \rightarrow \mathbb{Z}/p\mathbb{Z}$ is the canonical projection;
- (3) $I = I(F)$, in which case $I = \ker(\pi_2 \circ \text{sign}_p)$ for any $P \in X_F$.

In this section we will investigate in how far I determines N . In the first two cases (when $2 \notin I$) N is uniquely determined by I , but in the third case (when $2 \in I$) this is not so. We prove three auxiliary results first.

Lemma 5.1. *Let $P \in X_F$ and let $h \in W(A, \sigma)$ be such that $\text{sign}_P^H h = 0$. Then there exist $n \in \mathbb{N}_0$ and $q \in W(F)$ such that $\text{sign}_P q = 2^n$ and $q \cdot h \in W(A, \sigma)_t$, where $W(A, \sigma)_t$ denotes the torsion subgroup of $W(A, \sigma)$.*

Proof. Let $U = (\text{sign}^H h)^{-1}(0)$. The set U is clopen in X_F since the total signature map $\text{sign}^H h$ is continuous [1, 7.2]. Hence the function $\chi_U : X_F \rightarrow \mathbb{Z}$ defined by $\chi_U = 1$ on U and $\chi_U = 0$ on $X_F \setminus U$ is continuous. Therefore there are $n \in \mathbb{N}_0$ and $q \in W(F)$ such that $\text{sign } q = 2^n \chi_U$, cf. [9, VIII, 6.10]. It follows that $\text{sign}^H(q \cdot h) = \text{sign}(q) \text{sign}^H(h) = 0$ on X_F , and so $q \cdot h \in W(A, \sigma)_t$, by [11, 4.1]. ■

Lemma 5.2. *Let $(f, g) \in \text{Hom}_{(W(F), \mathbb{Z})}(W(A, \sigma), \mathbb{Z})$. Then there exists a unique $P \in X_F$ such that $f = \text{sign}_P$ and $\ker \text{sign}_P^H \subseteq \ker g$. In particular, if $P \in \text{Nil}[A, \sigma]$, then $g = 0$.*

Proof. The morphism of rings $f : W(F) \rightarrow \mathbb{Z}$ is completely determined by $P := \{a \in F^\times \mid f(\langle a \rangle) = 1\} \cup \{0\}$, which is an ordering of F (follow for instance the proof of [9, VIII, 3.4(2)]). Therefore $f = \text{sign}_P$.

Let $h \in \ker \text{sign}_P^H$. By (5.1) there exists $n \in \mathbb{N}_0$ and $q \in W(F)$ such that $\text{sign}_P q = 2^n$ and $q \cdot h \in W(A, \sigma)_t$. Since $f = \text{sign}_P$, we obtain $f(q) \neq 0$ and since g has values in \mathbb{Z} (which is torsion-free), we have $0 = g(q \cdot h) = f(q) \cdot g(h)$, which implies $g(h) = 0$. Therefore $\ker \text{sign}_P^H \subseteq \ker g$. ■

Lemma 5.3. *Let (I, N) be an m -ideal of $W(A, \sigma)$ with $N \neq W(A, \sigma)$ and with $I = \ker(\pi_p \circ \text{sign}_p)$ for some $P \in X_F$ and prime p . Then (I, N) is prime.*

Proof. Since I is a prime ideal of $W(F)$ we only need to check that for $q \in W(F)$ and $h \in W(A, \sigma)$ such that $q \cdot h \in N$ we have $q \in I$ or $h \in N$. Assume that $q \notin I$. Since $W(F)/I$ is a field, q is invertible modulo I , i.e. there exists $q' \in W(F)$ such that $qq' = 1 + i$ with $i \in I$. Hence the assumption $q \cdot h \in N$ implies $qq' \cdot h = (1 + i) \cdot h \in N$ and thus $h \in N$ since $i \cdot h \in I \cdot W(A, \sigma) \subseteq N$. ■

The case $2 \notin I$.

Lemma 5.4. *Let (I, N) be a prime m -ideal of the $W(F)$ -module $W(A, \sigma)$ such that $2 \notin I$. Then $W(A, \sigma)_t \subseteq N$.*

Proof. Let $h \in W(A, \sigma)_t$. Then there exists $n \in \mathbb{N}$ such that $0 = 2^n \cdot h$ by [15, 5.1]. Since $0 \in N$, this implies $2^n \in I$ or $h \in N$. Since $2 \notin I$, we obtain $h \in N$. ■

Proposition 5.5. *Let (I, N) be a prime m -ideal of the $W(F)$ -module $W(A, \sigma)$ with $2 \notin I$. Then one of the following holds:*

- (i) *There exists $P \in X_F$ such that $(I, N) = (\ker \text{sign}_P, \ker \text{sign}_P^H)$.*
- (ii) *There exist $P \in X_F$ and a prime $p > 2$ such that*

$$\begin{aligned} (I, N) &= (\ker(\pi_p \circ \text{sign}_P), \ker(\pi \circ \text{sign}_P^H)) \\ &= (p \cdot W(F) + \ker \text{sign}_P, p \cdot W(A, \sigma) + \ker \text{sign}_P^H), \end{aligned}$$

where $\pi : \text{Im sign}_P^H \rightarrow \text{Im sign}_P^H / (p \cdot \text{Im sign}_P^H)$ is the canonical projection.

Proof. We first observe that the ordering P we are looking for cannot be in $\text{Nil}[A, \sigma]$, since N is a proper submodule of $W(A, \sigma)$.

We denote by $\varphi : W(F) \rightarrow W(F)/I$ and $\psi : W(A, \sigma) \rightarrow W(A, \sigma)/N$ the canonical projections. By (4.8), $\text{Im } \psi$ is a torsion-free $\text{Im } \varphi$ -module, where the product is given by $\varphi(q) \cdot \psi(h) = \psi(q \cdot h)$ for $q \in W(F)$ and $h \in W(A, \sigma)$.

We consider the different possibilities for $I = \ker \varphi$. By the classification of prime ideals of $W(F)$ there is a $P \in X_F$ such that $\ker \text{sign}_P \subseteq I$. Let $h \in \ker \text{sign}_P^H$. By (5.1) there exists $n \in \mathbb{N}_0$ and $q \in W(F)$ such that $\text{sign}_P q = 2^n$ and $q \cdot h \in W(A, \sigma)_t$. Since $W(A, \sigma)_t \subseteq N$ by (5.4) we obtain $0 = \psi(q \cdot h) = \varphi(q) \cdot \psi(h)$. But we know that $I = \ker \text{sign}_P$ or $I = \ker(\pi_p \circ \text{sign}_P)$ for some prime $p > 2$. Since $\text{sign}_P(q) = 2^n$ and $p > 2$, we obtain that $q \notin I = \ker \varphi$. Thus $\varphi(q) \cdot \psi(h) = 0$ implies $\psi(h) = 0$ by (4.8). Therefore $\ker \text{sign}_P^H \subseteq N$.

We now consider the two possibilities for φ :

(i) $\ker \varphi = \ker \text{sign}_P$. Then $\text{Im } \varphi \cong \mathbb{Z}$ is a torsion-free abelian group, and it follows from (4.8) that $\text{Im } \psi$ is a torsion-free abelian group. Using $\ker \text{sign}_P^H \subseteq \ker \psi = N$, we obtain

$$\text{Im } \psi \cong W(A, \sigma)/N \cong (W(A, \sigma)/\ker \text{sign}_P^H)/(N/\ker \text{sign}_P^H),$$

where $W(A, \sigma)/\ker \text{sign}_P^H \cong \text{Im sign}_P^H$ is an infinite cyclic group. Since $\text{Im } \psi$ is torsion-free, the only possibility is $N/\ker \text{sign}_P^H = \{0\}$, i.e. $N = \ker \text{sign}_P^H$.

(ii) $\ker \varphi = \ker(\pi_p \circ \text{sign}_P)$. Then $\text{Im } \varphi$ is a field with p elements and $\text{Im } \psi \cong W(A, \sigma)/N$ is an $\text{Im } \varphi$ -vector space. In particular, for every $h \in W(A, \sigma)$, we have that

$p \cdot h \in N$. Since we also have $\ker \text{sign}_p^H \subseteq N$ we obtain $N_0 := p \cdot W(A, \sigma) + \ker \text{sign}_p^H \subseteq N$.

Consider an arbitrary element $p \cdot h + h_0 \in N_0$ with $h \in W(A, \sigma)$ and $h_0 \in \ker \text{sign}_p^H$. Then $\text{sign}_p^H(p \cdot h + h_0) = p \text{sign}_p^H h$ and sign_p^H induces an isomorphism $N_0 / \ker \text{sign}_p^H \cong p \cdot C$, where $C = \text{Im } \text{sign}_p^H$ is an infinite cyclic group. It follows that

$$W(A, \sigma) / N_0 \cong (W(A, \sigma) / \ker \text{sign}_p^H) / (N_0 / \ker \text{sign}_p^H) \cong C / (p \cdot C),$$

and so $[W(A, \sigma) : N_0] = p$.

Since $N / \ker \text{sign}_p^H$ is a subgroup of $W(A, \sigma) / \ker \text{sign}_p^H \cong C$, there must exist $k \in \mathbb{N}$ such that $N / \ker \text{sign}_p^H \cong k \cdot C$. It follows that

$$W(A, \sigma) / N \cong (W(A, \sigma) / \ker \text{sign}_p^H) / (N / \ker \text{sign}_p^H) \cong C / (k \cdot C),$$

which we know to be a vector space over a field with p elements. Thus every non-zero element of $W(A, \sigma) / N$ has order p and so $px \in k \cdot C$ for all $x \in C$. It follows that $k|p$. If $k = 1$, then $N = W(A, \sigma)$, a contradiction. Thus $k = p$ and so $[W(A, \sigma) : N] = p$.

We conclude that $N = N_0 = p \cdot W(A, \sigma) + \ker \text{sign}_p^H$.

Finally, consider the canonical projection $\pi : \text{Im } \text{sign}_p^H \rightarrow \text{Im } \text{sign}_p^H / (p \cdot \text{Im } \text{sign}_p^H)$. It is not difficult to see that $p \cdot W(A, \sigma) + \ker \text{sign}_p^H = \ker(\pi \circ \text{sign}_p^H)$. ■

The case $2 \in I$. Consider a prime m -ideal (I, N) of the $W(F)$ -module $W(A, \sigma)$ with $2 \in I$. As observed above we have $I = I(F)$, the fundamental ideal of $W(F)$.

We define $I(A, \sigma)$ to be the submodule of all equivalence classes in $W(A, \sigma)$ of forms of even rank, cf. [2, 2.2] or [4, §3.2.1]. Observe that if A is a division algebra, then $I(A, \sigma)$ is additively generated by the forms $\langle 1, a \rangle_\sigma$ for $a \in \text{Sym}(A, \sigma)^\times$ since every form can be diagonalized. Recall from [4, §3.2.1]:

Lemma 5.6. $I(A, \sigma)$ has index 2 in $W(A, \sigma)$.

Proof. Since the rank is invariant under Morita equivalence, we may assume that A is a division algebra, in which case the proof follows immediately from the fact that $\langle a \rangle_\sigma = \langle b \rangle_\sigma \pmod{I(A, \sigma)}$ for every $a, b \in \text{Sym}(A, \sigma)^\times$. ■

Proposition 5.7. A pair $(I(F), N)$ is a prime m -ideal of $W(A, \sigma)$ if and only if N is a proper submodule of $W(A, \sigma)$ with $I(F) \cdot W(A, \sigma) \subseteq N$.

Proof. Take $p = 2$ in (5.3). ■

We thus see that in contrast to the $2 \notin I$ case, N is not uniquely determined by I , since there are in general several proper submodules N of $W(A, \sigma)$ containing $I(F) \cdot W(A, \sigma)$. Obviously one such submodule is $I(F) \cdot W(A, \sigma)$ itself, and $I(A, \sigma)$ is another one. The following example shows that in general these modules are distinct.

Example 5.8. Let $F = \mathbb{R}((x))((y))$ be the iterated Laurent series field in the unknowns x and y over the field of real numbers. Let $D = (x, y)_F$ be the quaternion algebra with F -basis $(1, i, j, k)$ such that $i^2 = x$, $j^2 = y$ and $ij = k = -ji$. Let σ be the orthogonal involution on D that sends i to $-i$ and that fixes the other basis elements. Let $v : F \rightarrow \mathbb{Z} \times \mathbb{Z}$ be the standard (x, y) -adic valuation on F (see for instance [17,

Chapter 3]). Note that F is Henselian with respect to v . An application of Springer's theorem shows that the norm form $\langle 1, -x, -y, xy \rangle$ of D is anisotropic (we obtain four residue forms of dimension 1 over \mathbb{R} , that are necessarily anisotropic). Hence D is a division algebra, cf. [9, III, 4.8].

Since F is Henselian, the valuation v extends uniquely to a valuation on D (see [14, Thm. 2]), which we also denote by v . We now claim that the residue division algebra \overline{D} is isomorphic to \mathbb{R} . The proof of this claim goes as follows: We first compute Γ_D . Since $i^2 = x$ and $v(x) = (1, 0)$ we have $v(i) = (1/2, 0)$. Similarly $v(j) = (0, 1/2)$ and $v(k) = (1/2, 1/2)$. Let γ be the quaternion conjugation on D . Since v extends uniquely from F to D and $v \circ \gamma$ is a valuation on D we have $v(x) = v \circ \gamma(x)$ for every $x \in D$. In particular $v(\gamma(x)x) = 2v(x)$. If we write $x = a_0 + a_1i + a_2j + a_3k$ we obtain $\gamma(x)x = a_0^2 - xa_1^2 - ya_2^2 + xya_3^2$. Since the four terms in this sum have different valuations we get

$$\begin{aligned} v(x) &= \frac{1}{2} \min\{\varepsilon_0 0, \varepsilon_1 v(x), \varepsilon_2 v(y), \varepsilon_3 v(xy)\} \\ &= \frac{1}{2} \min\{\varepsilon_0 0, \varepsilon_1 (1, 0), \varepsilon_2 (0, 1), \varepsilon_3 (1, 1)\}, \end{aligned}$$

where $\varepsilon_i = 0$ if $a_i = 0$, and 1 otherwise, for $i = 0, \dots, 3$. This yields $\Gamma_D = \frac{1}{2}\mathbb{Z} \times \frac{1}{2}\mathbb{Z}$. In particular $[\Gamma_D : \Gamma_F] = 4$ and by Draxl's "Ostrowski Theorem" (see [6, Equation 1.2]), we obtain $[\overline{D} : \overline{F}] = 1$, i.e. $\overline{D} = \overline{F} = \mathbb{R}$. This proves the claim.

Now a system of representatives of $v(\text{Sym}(D, \sigma))$ is given by $\{(0, 0), (0, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2})\}$, corresponding to the set of uniformizers $\{1, j, k\}$, and by [10, Thm. 3.7] we obtain that

$$\begin{aligned} W(D, \sigma) &\cong W(\overline{D}, \text{id}) \oplus W(\overline{D}, \text{id}) \oplus W(\overline{D}, \text{id}) \\ &\cong W(\mathbb{R}) \oplus W(\mathbb{R}) \oplus W(\mathbb{R}) \\ &\cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}, \end{aligned} \tag{\star}$$

where the first isomorphism is obtained by computing the residue forms corresponding to the three uniformizers. We use this description of $W(D, \sigma)$ to compute $I(F) \cdot W(D, \sigma)$. Since $I(F)$ is additively generated by the forms $\langle 1, a \rangle$ for $a \in F^\times$ and $W(D, \sigma)$ is additively generated by the diagonal forms of dimension one (since D is division), we only have to consider the forms $\langle 1, a \rangle \cdot \langle \alpha \rangle_\sigma = \langle \alpha, a\alpha \rangle_\sigma$ for $a \in F^\times$ and $\alpha \in \text{Sym}(D, \sigma)^\times$. Since $v(a) \in \Gamma_F = \mathbb{Z} \times \mathbb{Z}$ we have $v(a\alpha) = v(a) + v(\alpha) \in v(\alpha) + 2\Gamma_D = v(\alpha) + \mathbb{Z} \times \mathbb{Z}$. It follows that the form $\langle \alpha, a\alpha \rangle_\sigma$ only gives rise to one residue form in $W(\overline{D}, \text{id}) \cong W(\mathbb{R}) \cong \mathbb{Z}$. This residue form has dimension 2 and belongs to $I(\mathbb{R}) \cong 2\mathbb{Z}$. Therefore $I(F) \cdot W(D, \sigma) \subseteq 2\mathbb{Z} \oplus 2\mathbb{Z} \oplus 2\mathbb{Z}$ (under the isomorphisms (\star)), so has index at least 8 in $W(D, \sigma)$. In particular $I(F) \cdot W(D, \sigma)$ cannot be equal to $I(D, \sigma)$ which has index 2 in $W(D, \sigma)$ by (5.6).

Since there may be several proper submodules of $W(A, \sigma)$ containing $I(F) \cdot W(A, \sigma)$, it is interesting to see if one of these submodules can be singled out by some natural property.

In the remainder of this paper we distinguish between hermitian forms h over (A, σ) and their classes $[h]$ in $W(A, \sigma)$. By the structure theory of F -algebras with involution and Morita theory, there exist $n \in \mathbb{N}$ and an F -division algebra with involution $(D, -)$ such that $W(A, \sigma) \cong W(M_n(D), -^t)$. Since rank modulo 2 is a Witt class invariant and since we will examine forms of even rank we may also assume for convenience of notation that $(A, \sigma) = (M_n(D), -^t)$.

For $a \in \text{Sym}(D, -)^\times$ we define the hermitian form

$$h_a : D^n \times D^n \longrightarrow M_n(D), (x, y) \longmapsto \bar{x}^t a y$$

(where x and y are considered as $1 \times n$ matrices), which is of rank one over (A, σ) . Let ξ be the bijection between hermitian forms over (A, σ) and hermitian forms over $(D, -)$ induced by Morita equivalence, as described in [12]. Then $\xi(h_a) = \langle a \rangle_-$.

For every $a \in \text{Sym}(A, \sigma)^\times$ we define a set of hermitian forms over (A, σ) by

$$\mathcal{F}_a := \{h_{a^{i_1}} \perp \cdots \perp h_{a^{i_\ell}} \mid \ell \in \mathbb{N} \text{ and } i_1, \dots, i_\ell \in \mathbb{N}_0\}.$$

Let $h_1 = h_{a^{i_1}} \perp \cdots \perp h_{a^{i_k}}$ and $h_2 = h_{a^{j_1}} \perp \cdots \perp h_{a^{j_\ell}}$ be in \mathcal{F}_a . We define

$$h_1 \boxtimes h_2 := \bigoplus_{p=1}^k \bigoplus_{q=1}^\ell h_{a^{i_p + j_q}},$$

which is again an element of \mathcal{F}_a .

Observe that $h_a \perp h_{-a}$ and $h_1 \perp h_{-a^2}$ are hyperbolic. Indeed, they are mapped by ξ to the forms $\langle a, -a \rangle_-$ and $\langle 1, -a^2 \rangle_- \cong \langle 1, -1 \rangle_-$ (since $a = \bar{a}$), respectively, which are both hyperbolic.

Observe also that $I(A, \sigma)$ is additively generated by the classes $[h_1 \perp h_a]$ for $a \in \text{Sym}(D, -)^\times$ since $\xi(I(A, \sigma)) = I(D, -)$.

Lemma 5.9. *Let $a \in \text{Sym}(D, -)^\times$ and let $h_1, h_2 \in \mathcal{F}_a$ be such that $[h_1 \boxtimes h_2] \in I(A, \sigma)$. Then $[h_1] \in I(A, \sigma)$ or $[h_2] \in I(A, \sigma)$.*

Proof. Write $h_1 = h_{a^{i_1}} \perp \cdots \perp h_{a^{i_k}}$ and $h_2 = h_{a^{j_1}} \perp \cdots \perp h_{a^{j_\ell}}$. Then $\text{rank } h_1 = k$, $\text{rank } h_2 = \ell$ and $\text{rank}(h_1 \boxtimes h_2) = k\ell$. Since $[h_1 \boxtimes h_2] \in I(A, \sigma)$ and Witt equivalence does not change the parity of the rank of forms we know that 2 divides $k\ell$ and the result follows. ■

Proposition 5.10. *Let (I, N) be a prime m -ideal of the $W(F)$ -module $W(A, \sigma)$ such that $2 \in I$. Then $N = I(A, \sigma)$ if and only if for every $a \in \text{Sym}(D, -)^\times$ and every $h_1, h_2 \in \mathcal{F}_a$ $[h_1 \boxtimes h_2] \in N$ implies $[h_1] \in N$ or $[h_2] \in N$.*

Proof. The necessary direction is (5.9). For the sufficient condition we will show that $I(A, \sigma) \subseteq N$, and the result will follow since $[W(A, \sigma) : I(A, \sigma)] = 2$ and N is a proper submodule of $W(A, \sigma)$. Let $a \in \text{Sym}(D, -)^\times$. Since $2 \in I$ we have $[2 \cdot h_a] \in N$, i.e. $[h_a] = [-h_a] = [h_{-a}] \pmod{N}$. Furthermore, $[(h_1 \perp h_a) \boxtimes (h_1 \perp h_{-a})] = [h_1 \perp h_{-a} \perp h_a \perp h_{-a^2}] = 0 \in N$. Therefore $[h_1 \perp h_a] \in N$ or $[h_1 \perp h_{-a}] \in N$, i.e. $[h_1 \perp h_a] \in N$ since $[h_a] = [h_{-a}] \pmod{N}$. Since $I(A, \sigma)$ is additively generated by the classes $[h_1 \perp h_a]$, we conclude that $I(A, \sigma) \subseteq N$. ■

6. CANONICAL IDENTIFICATION OF H -SIGNATURES AND MORPHISMS INTO \mathbb{Z}

Lemma 6.1. *Let $\rho, \tau : W(A, \sigma) \rightarrow \mathbb{Z}$ be two surjective morphisms of abelian groups such that $\ker \tau = \ker \rho$. Then there exists $\varepsilon \in \{-1, 1\}$ such that $\rho = \varepsilon \tau$.*

Proof. Let $N := \ker \tau$, and let $h \in W(A, \sigma)$ be such that $\tau(h) = 1$. Then $W(A, \sigma) = \mathbb{Z} \cdot h + N$, and $\rho(W(A, \sigma)) = \mathbb{Z} \cdot \rho(h)$. Since ρ is surjective we obtain $\rho(h) = \varepsilon \in \{-1, 1\}$. So for $h' = k \cdot h + n$ with $k \in \mathbb{N}_0$ and $n \in N$, we have $\tau(h') = k$ and $\rho(h') = \varepsilon k$. ■

Lemma 6.2. *Let (f, g) be a non-trivial $(W(F), \mathbb{Z})$ -morphism from $W(A, \sigma)$ to \mathbb{Z} . Then there exists $P \in X_F \setminus \text{Nil}[A, \sigma]$ such that (f, g) and $(\text{sign}_P, \text{sign}_P^H)$ are equivalent.*

Proof. As seen in the proof of (5.2), $f = \text{sign}_P$ for some $P \in X_F$. By (4.6) $(\ker f, \ker g)$ is a prime \mathfrak{m} -ideal of the $W(F)$ -module $W(A, \sigma)$. It then follows from (5.5) that $\ker \text{sign}_P^H = \ker g$. Since $g \neq 0$, $P \notin \text{Nil}[A, \sigma]$ and there is an isomorphism of abelian groups $\rho : \text{Im } g \rightarrow \mathbb{Z}$. Let $\tau : \text{Im } \text{sign}_P^H \rightarrow \mathbb{Z}$ be an isomorphism of abelian groups. Then $\rho \circ g$ and $\tau \circ \text{sign}_P^H$ are surjective morphisms of abelian groups from $W(A, \sigma)$ to \mathbb{Z} and by (6.1) there exists $\varepsilon \in \{-1, 1\}$ such that $\rho \circ g = \varepsilon(\tau \circ \text{sign}_P^H)$. Therefore $g = \vartheta \circ \text{sign}_P^H$, where $\vartheta = \varepsilon(\rho^{-1} \circ \tau) : \text{Im } \text{sign}_P^H \rightarrow \text{Im } g$ is an isomorphism of groups and therefore an isomorphism of \mathbb{Z} -modules. So (f, g) and $(\text{sign}_P, \text{sign}_P^H)$ are equivalent by (4.9). ■

As a consequence we obtain that H -signatures can be canonically identified with equivalence classes of $(W(F), \mathbb{Z})$ -morphisms from $W(A, \sigma)$ to \mathbb{Z} , more precisely:

Proposition 6.3. *There is a bijection between the pairs $(\text{sign}_P, \text{sign}_P^H)$ for $P \in X_F \setminus \text{Nil}[A, \sigma]$ and the equivalence classes of non-trivial $(W(F), \mathbb{Z})$ -morphisms of modules from $W(A, \sigma)$ to \mathbb{Z} , given by*

$$\begin{aligned} \{(\text{sign}_P, \text{sign}_P^H) \mid P \in X_F \setminus \text{Nil}[A, \sigma]\} &\longrightarrow \text{Hom}_{(W(F), \mathbb{Z})}^*(W(A, \sigma), \mathbb{Z})/\sim \\ (\text{sign}_P, \text{sign}_P^H) &\longmapsto (\text{sign}_P, \text{sign}_P^H)/\sim \end{aligned}$$

Proof. By (6.2) we know that this map is surjective, and it is injective since assuming $(\text{sign}_P, \text{sign}_P^H)/\sim = (\text{sign}_Q, \text{sign}_Q^H)/\sim$ implies $\text{sign}_P = \text{sign}_Q$, and thus $P = Q$. ■

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